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THE FORMATION OF VORTICES IN A FREE BOUNDARY LAYER

H. Schade and A. Michalke

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16. Abstract

Discussion of the phenomenon of ring-vortex formation for the case of low viscosity. The following explanation for the mechanism of formation is proposed: a wall boundary layer caused by friction is formed in the nozzle which produces the stream, with a free boundary layer forming in its wake; both can be calculated from boundary-layer theory. The boundary layer is unstable with respect to induction for a certain range of frequencies and wavelengths, which can be calculated from frictionless stability theory. All unstable wavelengths can be artificially excited; in the absence of artificial excitation, the occurring disturbance will be that of greatest possible instability. As soon as the dominant wavelength has established itself, the boundary layer rolls out because of the disturbed induction equilibrium, while the frequency and wavelength remain unchanged.

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THE FORMATION OF VORTICES IN A FREE BOUNDARY LAYER

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1. Introduction /147*

1.1. The Basic Phenomenon

It has long been known that the transition from a round laminar free jet to turbulence passes through a periodic intermediate state in which the cylindrical interface between the jet and the surrounding fluid rolls up in a meandering fashion. Figure 1 shows this process in water behind a nozzle with a diameter of 1.5 cm at a Reynolds number of 1800, based on average velocity in the nozzle and nozzle diameter; this photograph comes from a film by E. Berger [1] on the laminar-turbulent transition.

Fig. 1. Free jet in water at Reynolds number Re = 1800, nozzle diameter D = 1.5 cm.

As early as 1867, J. Tyndall [2] observed that the boundary of an air jet, made visible with smoke, is very unstable with respect to excitation by sound. A year later, H. Helmholtz [3], who was familiar with Tyndall's work and found confirmation for

*Numbers in the margin indicate pagination in the foreign text.

its results, concluded on the basis of theoretical considerations that the boundary would have to roll up. Although he does not expressly state how he arrived at this conclusion, it is reasonable to assume that he approximated the boundary, which is of course a cylindrical vortex sheet for an ideal fluid, by a series of discrete ring vortices and considered the induction effects these ring vortices have upon one another when they are deflected from the equilibrium position in the form of a slight waviness in the vortex sheet. Figure 2, top, shows a series of vortices arranged at the extrema and the inflection points of a sine curve. In addition, the velocities induced by the two /148 neighboring vortices at the location of a vortex and their resultant are plotted in each case. Below that, the same series of vortices is shown a brief time later, when each vortex has migrated a certain distance due to the induced velocity field; the velocities induced in the new location are also plotted again. The same configuration is plotted under this after another interval of the same length.

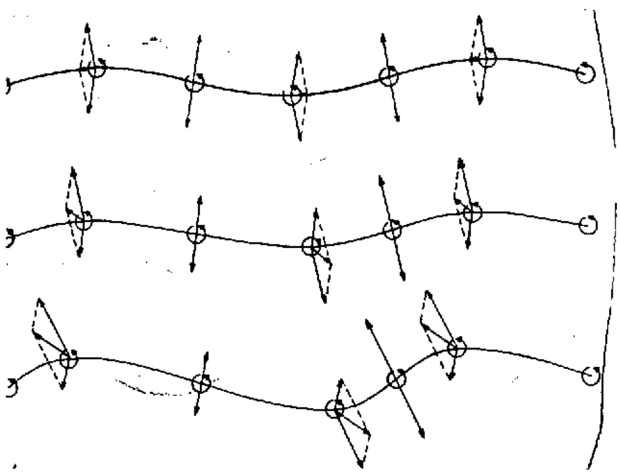


Fig. 2. The rolling up of a vortex sheet according to Helmholtz.

1.2. Early Theoretical Studies

The exact mathematical confirmation of this concept took place in two steps. Lord Rayleigh [4] was first able to demonstrate, in 1878, that a planar vortex sheet in an ideal fluid is unstable vis-a-vis small wavy disturbances of any given frequency and wavelength. In 1931, L. Rosenhead [5] then demonstrated that a planar vortex sheet in an ideal fluid rolls up in a meandering fashion after it is once perturbed in a wavy configuration. Rosenhead's calculations were given a critical check in 1959 by G. Birkhoff and J. Fisher [6] and were then improved, independently of this, by F.R. Hama and E.R. Burke [7].

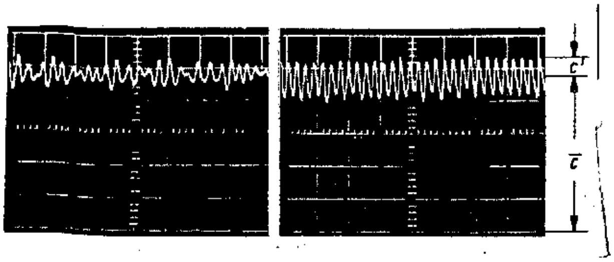
The following considerations are essential in this work: Rayleigh's calculations do demonstrate the instability of the vortex sheet but can not explain why only one of the infinite number of unstable wavelengths in the experiment occurs such that an observable phenomenon takes place. This is a result of neglecting viscosity: The vortex sheet is unstable because the induction effects resulting from the various elementary vortices are in unstable equilibrium. This is explainable within the theory of ideal fluids, and the addition of fluid friction has only a damping effect on this[1]. The favoring of a specific frequency and wavelength is a consequence of the shear profile which develops under the influence of viscosity, however. This is directly evident for a planar vortex sheet; in this case, a characteristic length manifests itself only as the result of taking the shear profile into consideration, and this is the condition for the occurrence of a special frequency or wavelength. For a cylindrical vortex sheet, we could consider jet diameter to be a characteristic length; the good agreement between the data obtained with the round free jet and the theoretical results derived for the planar case indicates, however, that the effect of axial symmetry is secondary. This is also reasonable because a phenomenon is involved in which the boundary layer is small relative to jet diameter. Studies on the effect of axial symmetry are now in progress.

[1]In contrast, it is known that Blasius' laminar boundary layer, for instance, becomes unstable only when friction is taken into consideration. It is desirable to designate these two types of unstable flow as induction-unstable and friction-unstable, respectively.

1.3. Previous Experimental Studies

The vortices produced in the axially symmetrical free jet are being studied at the DVL's Institut für Turbulenzforschung in a series of projects with the aid of the hot-wire measurement method. The first results of these studies can be found in two reports by R. Wille, U. Domm, H. Fabian and O. Wehrmann [8] and by R. Wille, O. Wehrmann and H. Fabian [9]. H. Fabian [10] studied the velocity profile of the vortex rings, while O. Wehrmann [11] concerned himself with acoustic effects on vortex rings. O. Wehrmann and R. Wille [12] and R. Wille [13] provide a summary of the studies. Independently, H. Sato [14, 15] has applied similar studies to a free shear layer behind a step and to a planar free jet.

Measurements of two quantities are essential for this work: First, we can measure the frequency of vortices which develop without artificially influencing the flow. Studies by O. Wehrmann [16] have indicated that this natural vortex development behind the nozzle is not strictly periodic, but rather that the frequency fluctuates about a certain mean. A timewise mean for this so-called natural frequency can be determined from a /149 large number of frequency measurements. On the other hand, we can artificially impress a sound field on the jet of such low intensity that it is not picked up by the hot-wire probe. Vortex formation at this frequency can then be induced in a region on both sides of the natural frequency at a point in the flow where vortices cannot be observed without the sound field. Figure 3 shows the natural and artificially excited velocity fluctuations, made visible on an oscillograph, which are caused by the vortices which pass by the hot-wire probe.



Natural Vortex Frequency

Vortex Frequency Excited by Sound

Fig. 3. Oscillograms of a) the natural vortex frequency, and b) the frequency excited by sound. U_0 = 4.53 m/sec, D = 10 cm, x/D = 0.38.

Due to the great deal of regularity in the artificially excited vortices, it is also possible in this case to measure the distance between two vortices from the phase relationship between a fixed and a movable hot-wire probe.

1.4. Subject of This Study

The process of vortex formation could thus be described as follows: Due to friction, a wall boundary layer forms in the nozzle and a free shear layer downstream from it which can be computed at low viscosity with the aid of boundary layer theory. At low viscosity, this shear layer is induction-unstable with respect to a certain range of frequencies and wavelengths, which can be calculated with the aid of stability theory -- at the very low viscosity, with the aid of the frictionless stability theory to a good approximation. All unstable frequencies and the associated wavelengths can be excited by artificial stimulus; without artificial effects, the most unstable disturbance primarily develops. Once a dominant wavelength has developed, the wave rolls up, at low viscosity, under the influence of the mutual induction of elementary vortices, maintaining frequency and wavelength, as Rosenhead described for an ideal fluid.

The subject of this work is a first attempt to calculate the natural frequency derived from this mechanism and the functional relationship between the exciting frequency and the wavelength which develops, and to compare these with the empirical results.

2. Velocity Field

2.1. Steady Region

A study will be made to determine the quantities upon which the velocity at a point in the flow depends. First of all, it of course depends upon the geometry of the nozzle which produces the jet. Since this relationship is of no interest in the present context, it can be eliminated by using only geometrically similar nozzles: the vortex filament nozzles described by A. Michalke [17] were employed, which have the advantage that the potential velocity field therein can be calculated relatively easily. Figure 4 shows the contour of such a vortex filament nozzle.

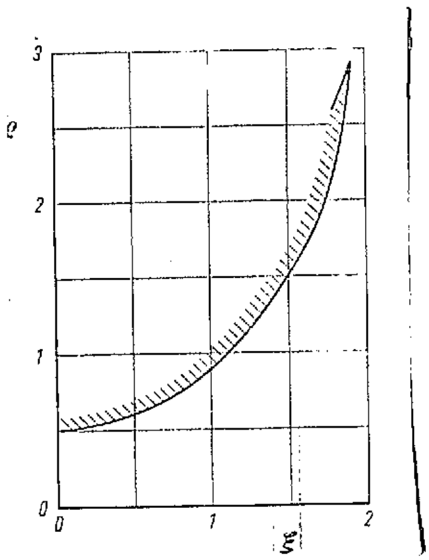


Fig. 4. Wall contour of the vortex filament nozzle. ρ, ξ = dimensionless cylindrical coordinates.

Aside from this, the magnitude of velocity at a point in the flow is a function of five variables:

Two position coordinates in the meridian plane, such as working length x measured along the line of wall flow in the nozzle and along its continuation in the free jet, and a coordinate y perpendicular to it;

A characteristic length for the nozzle, such as its diameter D at the end cross section[2];

[2]The selection of nozzle diameter as a characteristic length could be taken as a contradiction to the statement made above that axial symmetry is only of secondary importance. This objection is

A characteristic velocity, such as the velocity U_o on the nozzle axis at the end cross section;

The kinematic viscosity ν of the fluid. If the units of length and velocity are given, three independent parameters remain, e.g. the two position coordinates x/D and y/D and viscosity ν/U_oD or Reynolds number

$$Re = U_oD/\nu \quad (1)$$

If we take Prandtl's boundary layer equations as our basis, rather than the Navier-Stokes equations, we can also eliminate the dependence upon Reynolds number by means of the well-known variable transformation $\eta = (y/D)\sqrt{Re}$. Velocity at a given point in the flow can then be written in the form

$$\frac{u}{U_0} = \frac{u}{U_0}\left(\frac{x}{D}, \frac{y}{D}\sqrt{Re}\right)$$

i.e. we can describe the velocity field in and downstream from /150
such a nozzle by one velocity profile

$$\frac{u}{U_0} = \frac{u}{U_0}\left(\frac{y}{D}\sqrt{Re}\right)$$

each for each value of working length x/D.

The complete computation of these profiles is so complex that we consider it sufficient to characterize each by two characteristic quantities. Within the nozzle, velocity U_p at the margin of the boundary layer and displacement thickness

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2 (cont'd) eliminated if we perhaps selected the length of the nozzle contour instead. For geometrically similar nozzles, however, all lengths are proportional and therefore of equivalent value as reference lengths, and the diameter of the nozzle at the end cross section is the most graphic.

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U_p}\right) dy \qquad (2)$$

can be used for this purpose. Figure 5 shows the curves for these two characteristic quantities from studies by A. Michalke [17]. Computation of the velocity profiles outside the nozzle is still in progress.

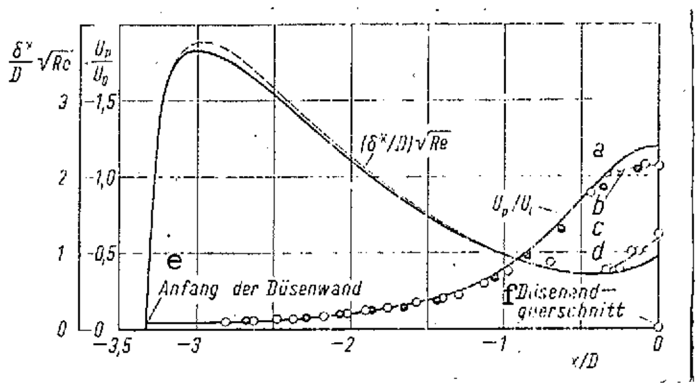

Fig. 5. Displacement thickness $(\delta^*/D)\sqrt{Re}$ of the boundary layer and potential velocity U_p/U_0 along the wall of the vortex filament nozzle (x/D arc length, U_0 velocity at center of end nozzle cross section).

Key:

a. Theoretical curve of potential velocity
b. Curve of potential velocity corrected on the basis of measurements
c. Displacement thickness of boundary layer, calculated on the basis of curve b
d. Displacement thickness of boundary layer, calculated on the basis of curve a
e. Beginning of nozzle wall
f. End cross section of nozzle

—— Modified Kotschin method
--- Zaat method
O Data for D = 10 cm
● Data for D = 7.5 cm
[Commas in numerals = decimal points.]

The velocity profiles inside and outside the nozzle have also been measured by a method described by O. Wehrmann [16] and A. Michalke [17]. Figure 6 shows two of the profiles plotted by a light-spot recorder at the end cross section of the nozzle and a short distance downstream. Average profiles were plotted from measurements at various velocities; these are shown in Fig. 7. In the end cross section of the nozzle, the two characteristic quantities were found to be

$$U_p/U_o = 1.06, \quad (\delta^*/D)\sqrt{Re} = 1.20 \qquad (3)$$

2.2. Periodically Nonsteady Region

The addition of dependence upon time makes the velocity field so complex that it is reasonable to limit ourselves from the outset to a consideration of two conveniently measurable characteristics of flow in this region, namely the frequencies and wavelengths of the resultant vortices.

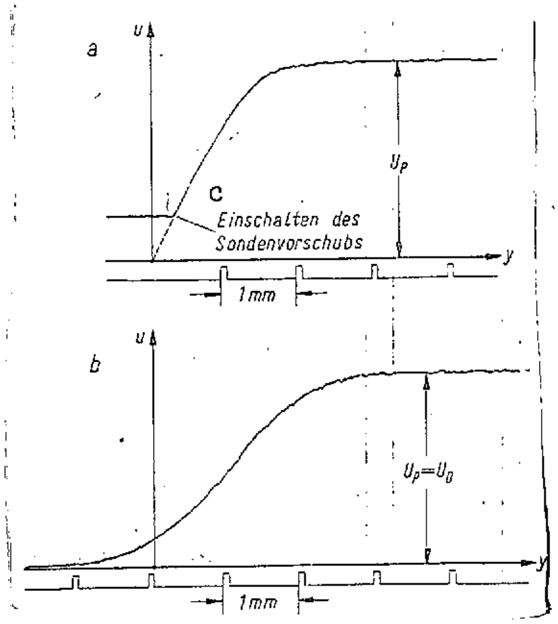

Fig. 6. Two velocity profiles in and downstream from the vortex filament nozzle, plotted by a light spot recorder.

a) D = 10 cm, U_o = 4.53 m/sec, x = -0.2 mm

b) D = 10 cm, U_o = 4.53 m/sec, x = 16 mm

Key: c. Probe advance started

If the above-stated assumption that viscosity is not of direct importance to the rolling-up process, but is only of secondary importance via the formation of a shear profile, is valid, these parameters depend only upon a characteristic length and a characteristic velocity for the shear profile, but not also upon the viscosity of the medium, if consideration is limited to vortex filament nozzles. U_p and δ^* in the nozzle end cross section can, according to the above considerations, serve as these characteristic quantities; since U_p and U_o have a constant ratio there, regardless of the experimental conditions, it is reasonable to use the more easily measured U_o in place of U_p. Accordingly,

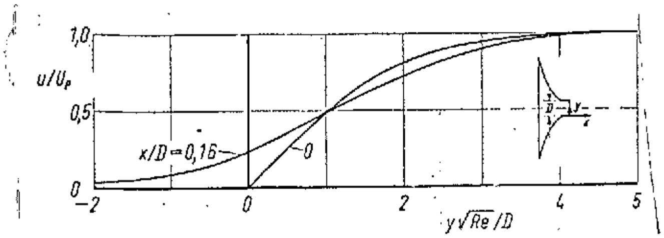

Fig. 7. Averaged boundary layer profiles in and downstream from the vortex filament nozzle. U_p = = potential velocity, diameter D = 10 cm.

the frequency $f\delta^*/U_0$ and the wavelength λ/δ^* of natural perturbations would have to be constants independent of Reynolds number, and the functional relationship between frequency and wavelength in artificially excited vortices would have to be representable by a single curve independent of Reynolds number.

It is evident that this will not be the case at very high viscosity or low Reynolds number: Friction would then have a damping action, and flow in the nozzle and in the free jet could no longer be calculated on the basis of boundary layer theory, i.e., the dimensional considerations formulated in the /151
application of boundary layer theory no longer apply, for the axial symmetry of the problem can certainly no longer be neglected. For large Reynolds numbers, however, these concepts have been confirmed not only by H. Sato [14, 15] for the separated boundary layer behind a step and for the planar free jet, but also by the studies published here for the round free jet. In addition, they also explain the rule, observed by H. Sato [14, 15] and O. Wehrmann [16], that the natural frequency is proportional to the 3/2 power of velocity; at most,

$$\frac{f\delta^*}{U_0} \sim \frac{fD}{U_0\sqrt{Re}} \sim f D^{1/2} \nu^{1/2} U_0^{-3/2} \qquad (4)$$

can still be dependent upon nozzle contour.

Figure 8 shows dimensionless natural vortex frequency, averaged over time, as a function of Reynolds number for vortex filament nozzles with diameters of D = 10 cm and D = 7.5 cm. In addition, results are plotted from hot-wire measurements in water downstream from a vortex filament nozzle with a diameter of D = 1.5 cm. We obtain

$$f\delta^*/U_0 = 0.0234 \tag{5}$$

as the average for natural frequency.

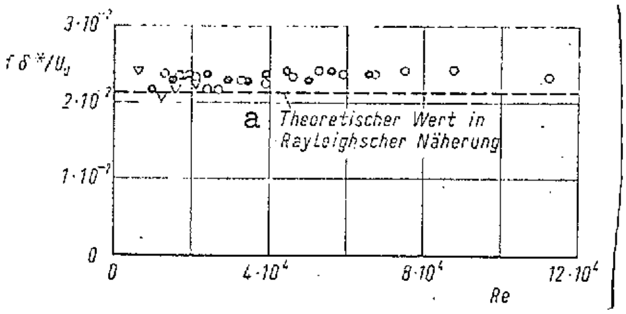

Fig. 8. Natural vortex frequency $f\delta^*/U_0$ downstream from a vortex filament nozzle as a function of Reynolds number Re = U_0D/ν.

δ^* = Displacement thickness of boundary layer in nozzle end cross section;

Measurements in air for nozzle diameter D = 10 cm (o), D = 7.5 cm (•), and in water with D = = 1.5 cm (∇).

Key: a. Theoretical value in Rayleigh's approximation

O. Wehrmann [16] has measured the distance between two successive vortices under artificial excitation as a function of frequency for three different velocities. One obtains the wavelength associated with each vortex frequency in this manner. Figure 9 shows the experimental results plotted in dimensionless form. From this diagram we find the wavelength associated with the natural frequency to be

$$\lambda/\delta^* = 25.1. \tag{6}$$

3. Stability Study

3.1. Nozzle Flow

The question arises as to what velocity profile is responsible for instability. It was probable from the outset that it would be a profile measured a short distance ahead of the

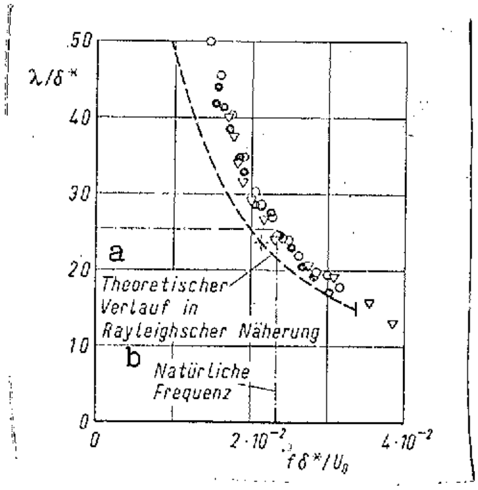

Fig. 9. Wavelength λ/δ^* as a function of frequency $f\delta^*/U_0$ downstream from the vortex filament nozzle; nozzle diameter D = 10 cm, δ^* = displacement thickness of boundary layer in nozzle end cross section, velocity U_0 = 7.3 m/sec (o), U_0 = 6.0 m/sec (●), U_0 = 4.5 m/sec (∇).

Key: a. Theoretical curve in Rayleigh's approximation
b. Natural frequency

observed rolling-up process. Nevertheless, a check was first made as to whether the boundary layer profile might already be unstable in the nozzle, based on stability theory. Since, according to Fig. 5, boundary layer thickness decreases along a large portion of the nozzle contour -- this is also plausible on the basis of the highly accelerated flow -- and only increases again in the last part, where acceleration approaches zero, the profile in the end cross section was studied with respect to its stability. The stability curves obtained by T. Tatsumi [18] for pipe inlet flow were taken as a basis. This appeared justified because the velocity profile in the end cross section of the nozzle has a distinct boundary layer character, and we know that boundary layer flows can be considered parallel on the basis of the stability theory[3].

Proceeding from Prandtl's boundary layer equations for flows whose velocity field, in cylindrical coordinates, is of the form $\{v_r(r,z), 0, v_z(r,z)\}$, T. Tatsumi calculates a series

[3] In the stability theory, those flows are called parallel flows in which the velocity vectors are parallel to one another at all points in the flow field.

of velocity profiles for pipe inlet flow in the first part of his article. In place of cylindrical coordinates z and r, he introduces the new dimensionless variables

$$X = \frac{z}{aR}, \quad Y = \frac{1}{2}\left[1 - \left(\frac{r}{a}\right)^2\right] \tag{7}$$

for this purpose; here a is pipe radius and R is a Reynolds number based on the velocity averaged over the pipe cross section and upon pipe radius. He characterizes the velocity profiles for various values of X by means of a displacement thickness

$$\delta(X) = \int_0^{1/2} \left[1 - \frac{u(X,Y)}{U_0(X)}\right] dY, \tag{8}$$

where u(X,T) is the axial velocity component at point (X,T), and U_0(X) is velocity on the axis, i.e. at the point (X, 1/2).[4] From the inlet to the Hagen-Poiseuille profile, δ increases from 0 to 0.25. Boundary layer theory is of course only applicable as long as a flow core exists; in this region, δ varies between 0 and 0.15. The boundary layer profiles u/U_0 for various X or δ can be made to coincide if we introduce the variable

$$H = T/\delta \tag{9}$$

in place of T. The value of boundary layer thickness δ which would be associated with the profile in the nozzle end cross section is of course a function of Reynolds number Re = $U_0 D/\nu$. /152 Table 1 contains various pairs of values; the fact that the values of δ for Re > 10^3 all lie below 0.15 basically just confirms the fact that flow in the nozzle's end cross section has

[4]For pipe inlet flow, this velocity is practically equal to the velocity U_p(X) at the margin of the boundary layer, since the lines of flow are not noticeably bent.

boundary layer character in this Reynolds number range.

TABLE 1.

$\frac{U_0 D}{\nu}$	δ	$\left(\frac{U_0 D}{\nu}\right)_{crit}$
10^3	0,9	$2,8 \cdot 10^4$
10^4	0,025	$4,0 \cdot 10^4$
10^5	0,009	$1,2 \cdot 10^5$
10^6	0,0025	$3,0 \cdot 10^5$

In the second part of his article, T. Tatsumi derives the axially symmetrical counterpart to the Orr-Sommerfeld equation[5] first given by J. Pretsch [19], apparently without being acquainted with Pretsch's article. If we write the differential equation for perturbation as a function of H, we merely have a velocity profile for all values of δ, but then δ occurs explicitly in the differential equation, so we still obtain different stability curves for different values of δ. These curves were calculated by T. Tatsumi for several values of δ.

The critical Reynolds numbers for the δ values in Table 1 have been determined by interpolation from the values given by T. Tatsumi. We see that up to Reynolds numbers of about 10^5, the critical Reynolds number is larger than the actual Reynolds number for the flow, i.e., flow is stable. This corresponds at the same time to the limit above which no measurements were made because flow became too turbulent.

3.2. Free Jet

Further study was based on the velocity profile for x/D = = 0.16. That is the last profile measured, since the rolling-up process becomes appreciable just downstream from this in the jet.

[5] Cf. footnote [6].

The surprising result was that the stability calculation even for a quite rough approximation of this profile agreed quite well with the measurement results, i.e. when we simply replace it with a straight line whose slope coincides with that of the measured profile at half its height (cf. Fig. 10).

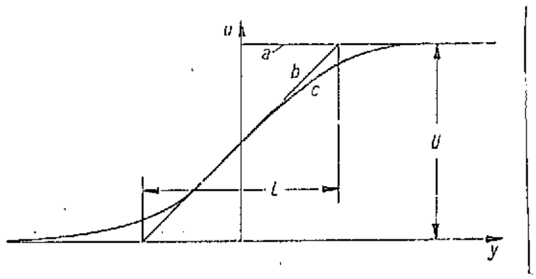

Fig. 10. Approximation of the measured velocity profile in the free jet; a. first approximation, b. second approximation, c. measured profile.

The frictionless stability calculation for this profile was published as early as 1880 by Lord Rayleigh [20]. He employs Euler's equations in Cartesian coordinates using

$$\left.\begin{aligned} u = \overline{u}(y) + \tilde{u}(x,y,t), \quad v = \tilde{v}(x,y,t), \quad w = 0 \\ q = \overline{q}(x) + \tilde{q}(x,y,t) \end{aligned}\right\} \tag{10}$$

for the velocity components and for pressure divided by the constant density; the quantities with bars are meant to represent the basic flow, and the quantities with tildes are meant to represent a small perturbation in this basic flow. Since the basic flow must satisfy Euler's equations, the terms which are free of perturbation quantities cancel out. By neglecting the quadratic terms in the perturbation quantities, moreover, Lord Rayleigh obtains the so-called linearized frictionless differential perturbation equations

$$\left.\begin{aligned} \frac{\partial \tilde{u}}{\partial t} + \overline{u}\frac{\partial \tilde{u}}{\partial x} + \overline{u}'\tilde{u} &= -\frac{\partial \tilde{q}}{\partial x}, \\ \frac{\partial \tilde{v}}{\partial t} + \overline{u}\frac{\partial \tilde{v}}{\partial x} &= -\frac{\partial \tilde{q}}{\partial y}, \\ \frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} &= 0. \end{aligned}\right\} \tag{11}$$

We see that we can satisfy these equations with the expression

$$\left.\begin{aligned}\tilde{v} = \varphi(y)\, e^{i\alpha(x-ct)}, \quad \tilde{u} = \frac{i}{\alpha}\varphi'(y)\, e^{i\alpha(x-ct)}, \\ \tilde{q} = -i\alpha\, e^{i\alpha(x-ct)} \int (\overline{u} - c)\,\varphi\, dy\end{aligned}\right| \tag{12}$$

where ϕ must satisfy Rayleigh's equation[6]

$$(\overline{u} - c)(\phi'' - \alpha^2\phi) - \overline{u}''\phi = 0 \tag{13}$$

The theory of this equation is generally quite complex and lacking in clarity; but for a velocity profile such as that in the present case, composed of linear subsections, it is possible to specify the general solution immediately; it has already been discussed by Lord Rayleigh and has the form

$$\phi = Ae^{\alpha y} + Be^{-\alpha y} \tag{14}$$

where integration constants A and B are different in each region of the velocity profile. Since $\tilde{u}$ and $\tilde{v}$ must vanish at infinity, we obtain the following more detailed form in the case at hand (cf. Fig. 10):

$$\varphi = \left\{\begin{array}{lll} A_1 e^{\alpha y} & \text{for} & y < -\frac{L}{2}, \\ A_2 e^{\alpha y} + B_2 e^{-\alpha y} & \text{for} & -\frac{L}{2} < y < \frac{L}{2}, \\ B_3 e^{-\alpha y} & \text{for} & \frac{L}{2} < y. \end{array}\right| \tag{15}$$

[6] Proceeding from the Navier-Stokes equations, first W. McF. Orr [21], for a special case, and then A. Sommerfeld [22], in a complete generalization, derived the corresponding differential equation for real fluids. This equation was first solved successfully by W. Tollmien [23] and H. Schlichting [24, 25, 26]; their theory was later developed primarily by C.C. Lin [27] and can now be considered complete; cf. [28].

Boundary conditions to be applied are that $\tilde{v}$ and $\tilde{q}$ be continuous at the interfaces. If we substitute (14) into (12), we find that ϕ and $(\bar{u} - c)\phi' - \bar{u}'\phi$ are continuous as we pass through the interfaces. This provides four homogeneous linear equations for the four integration constants A_1, A_2, B_1 and B_2; this system only has nontrivial solutions if its coefficient determinants /152 vanish. This condition leads to the equation

$$\frac{c}{U} = \frac{1}{2} \pm \frac{1}{2\alpha L} \sqrt{(\alpha L - 1)^2 - e^{-2\alpha L}}, \tag{16}$$

where the significance of U and L can again be seen from Fig. 10. For $L \to 0$, this formula naturally yields the result $c/U = (1 \pm i)/2$ for the vortex layer. This equation assigns to each dimensionless wave number αL or, with $\alpha = 2\pi/\lambda$, to each dimensionless wavelength λ/L a generally complex dimensionless velocity c/U and thus also a dimensionless frequency $fL/U = c_r L/\lambda U$. The profile is unstable relative to all wavy perturbations for which an imaginary part of c exists. This is the case for $\alpha L - 1 - e^{-\alpha L} < 0$ or for $\alpha L < 1.278$ and $\lambda/L > 4.916$. The associated frequency range $fL/U < 0.1017$ would have to correspond to the region in which artificial vortices can be produced. The factor for timewise amplification is αc_i; we obtain the factor for spatial amplification if we move with the perturbation wave, i.e. substitute $x = c_r t$; it is thus $\alpha c_i/c_r$. Both are zero at the boundaries of the unstable region and have a minimum at a point between them. Since the dimensionless propagation velocity c_r/U is constant over the entire unstable region (= 0.5), this occurs for both at the same point $\alpha L = 0.797$ or $\lambda/L = 7.884$, where $\alpha L c_i/U = 0.20118$ and $\alpha L c_i/c_r = 0.40237$. The associated frequency is $fl/U = 0.06342$. This point of maximum amplification corresponds to natural perturbation in the experiment.

If we now lay this profile on the measured profile in such a manner that they are tangent to one another at the half-height point, we obtain $U = U_o$ and $L = 3\delta^*$ as the relationship between the characteristic quantities for the two profiles. For the unstable region we thus obtain $f\delta^*/U_o < 0.0339$, and for the natural frequency and wavelength we obtain

$$f\delta^*/U_o = 0.0211 \text{ and } \lambda/\delta^* = 23.7 \qquad (17)$$

These values and the functional relationship between frequency and wavelength have been included in Figs. 8 and 9.

M. Lessen and J.A. Fox [29, 30] have carried out a stability calculation for a profile which, similar to a boundary layer, develops from a vortex layer. To all appearances, this profile represents a considerably better approximation to the measured profile. Since we know how sensitive stability calculations are relative to changes in the curve of the second derivative of the profile, we could suppose that the results of the stability calculation for Lessen's profile agree considerably better with the measurement results. This is not the case, however. We shall therefore not attempt a detailed comparison with the Lessen-Fox theory here.

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